Numerically generated quasi-equilibrium orbits of black holes: Circular or eccentric?

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We make a comparison between results from numerically generated, quasi-equilibrium configurations of compact binary systems of black holes in close orbits, and results from the post-Newtonian approximation. The post-Newtonian results are accurate through third PN order ($O(v/c)^6$ beyond Newtonian gravity), and include rotational and spin-orbit effects, but are generalized to permit orbits of non-zero eccentricity. Both treatments ignore gravitational radiation reaction. The energy E and angular momentum J of a given configuration are compared between the two methods as a function of the orbital angular frequency Ω. For small Ω, corresponding to orbital separations a factor of two larger than that of the innermost stable orbit, we find that, if the orbit is permitted to be slightly eccentric, with e ranging from ≈ 0.03 to ≈ 0.05, and with the two objects initially located at the orbital apocenter (maximum separation), our PN formulae give much better fits to the numerically generated data than do any circular-orbit PN methods, including various "effective one-body" resummation techniques. We speculate that the approximations made in solving the initial value equations of general relativity numerically may introduce a spurious eccentricity into the orbits.

I. INTRODUCTION AND SUMMARY

The late stage of inspiral of binary systems of neutron stars or black holes is of great current interest, both as a challenge for numerical relativity, and as a possible source of gravitational waves detectable by laser interferometric antennas. Because this stage, corresponding to the final few orbits and ultimate merger of the two objects into one, is highly dynamical and involves strong gravitational fields, it must be handled by numerical relativity, which attempts to solve the full Einstein equations on computers (see Refs. [1, 2] for reviews).

The early stage of inspiral can be handled accurately using post-Newtonian techniques, which involve an expansion of solutions of Einstein's equations in powers of $\epsilon \sim (v/c)^2 \sim Gm/rc^2$, where v, m, and r are typical velocity, mass and separation in the system, respectively. By expanding to very high powers of ϵ, increasingly accurate formulae can be derived to describe both the orbital motion and the gravitational waveform. Currently, results accurate through 3.5PN order ($O(\epsilon^{7/2})$ beyond Newtonian gravity) are known [3, 4, 5, 6, 7, 8, 9].

An important issue in understanding the full inspiral of compact binaries is how to connect the PN regime to the numerical regime. This is a non-trivial issue because the PN approximation gets worse the smaller the separation between the bodies. On the other hand, because of limited computational resources, numerical simulations cannot always be started with separations sufficiently large to overlap the PN regime where it is believed to be reliable.

Numerical simulations of compact binary inspiral start with a solution of the initial value (I-V) equations of Einstein's theory; these provide the initial data for the evolution equations (some simulations [10] solve in addition one of the six dynamical field equations). The initial state is assumed to consist of two compact objects (neutron stars or black holes) in an initially circular orbit. The circular-orbit condition is imposed by demanding that $dr/dt = 0$ initially, where r is a measure of the orbital separation. More precisely, the system is presumed to have an initial "helical Killing vector" (HKV), which corresponds to a kind of rigid rotation of the binary system. This amounts to ignoring initially the effects of gravitational radiation reaction. It also implies that the black holes are co-rotating, a condition which is astrophysically unlikely, albeit computationally advantageous. To make the problem tractable numerically, it is also generally assumed that the spatial metric is conformally flat. This approximation is usually justified by the neglect of radiation reaction in the initial state. For black hole binaries, suitable horizon boundary conditions must be imposed, while for neutron star binaries, hydrodynamical equations and an equation of state must be provided.

One important product of these initial value solutions is a relationship between the energy E and angular momentum J of the system as measured at infinity, and the orbital frequency Ω. As all quantities are well-defined and gauge invariant, they are useful variables for making

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comparisons with PN methods.

We have developed a formula for $E(\Omega)$ and $J(\Omega)$ using PN methods. Our analytic formula includes point-mass terms through 3PN order, but ignores radiation reaction. It also includes rotational energy and spin-orbit terms for the case in which the bodies are rotating. For black holes, tidal effects can be ignored. In contrast to previous work [11, 12], our formulae apply to general eccentric orbits, not just to circular orbits.

We then compare this formula with HKV numerical solutions for corotating binary black holes obtained by Grandclément et al. [10], for the regime where the black holes are separated from the location of the innermost circular orbit by a factor of around two, where PN results might be expected to work well ($Gm/rc^2 \sim 0.1$). We find the following two results:

1. When we assume circular PN orbits, our PN formulae for $E(\Omega)$ and $J(\Omega)$ agree well with other PN methods, including those using resummation or Padé techniques. However all PN methods consistently and systematically underestimate the binding energy and overestimate the angular momentum, compared to the values derived from the numerical HKV simulations, by amounts that are larger than the spread among the PN methods.
2. When we relax the assumption of a circular orbit and demand only that $dr/dt = 0$, our PN formula agrees extremely well with the numerical data. In this case the system is found to be initially at the apocenter of a slightly eccentric orbit. For values of $Gm\Omega/c^3$ ranging from 0.03 to 0.06, corresponding to orbital v/c between 0.3 and 0.4, or orbital separation between 10 and 6 Gm/c^2, perfect agreement with the binding energy can be obtained with eccentricities that range from 0.03 to 0.05.

If this apparent eccentricity is a real effect, there are two possible explanations. One is that the imposition of a helical Killing vector, which is equivalent to assuming a stationary binary star configuration as seen in a rotating frame, is not sufficient to guarantee a circular orbit in the relativistic regime. In Newtonian gravity, this assumption would be equivalent to imposing $dr/dt = 0$, which only puts the orbit at a turning point. An additional condition $d^2r/dt^2 = 0$ is needed to enforce a circular orbit. But this is a dynamical statement, which is outside the realm of the I-V equations of Einstein's theory. It may be that, in solving one of the dynamical field equations (the trace of the equation for $\partial K^{ij}/\partial t$) in addition to the four I-V equations, Grandclément et al. take care of this automatically.

The more likely possibility is that the approximations made in most numerical solutions, the main one being that the spatial metric is conformally flat, somehow introduce a systematic eccentricity into the orbit.

At present, these results can only be considered a hint of a possible eccentricity, however. It is entirely possible that circular PN orbits *are* consistent with the HKV results within the errors of the numerical simulations. This can only be decided if and when the numerical groups that carry out these simulations publish quantitative error bars determined by studying the convergence properties of the solutions as a function of grid size, domain size and other computational assumptions.

The remainder of this paper sketches the arguments that lead to these conclusions. Detailed derivations and formulae, and applications to neutron-star simulations, will be the subject of a future publication.

II. ORBITS AT THE TURNING POINT IN POST-NEWTONIAN GRAVITY

In Newtonian gravity, the orbit of a pair of point masses may be described by the set of equations

$$\begin{aligned} p/r &= 1 + e\cos(\phi-\omega)\,, \\ r^2\Omega &\equiv r^2 d\phi/dt = (mp)^{1/2}\,, \\ E &= \mu(\dot r^2 + r^2\Omega^2)/2 - \mu m/r\,, \\ J &= \mu|\mathbf{x}\times\mathbf{v}|\,, \end{aligned} \tag{1}$$

where $p = a(1-e^2)$ is the semi-latus rectum (a is the semi-major axis), ω is the angle of pericenter, $m = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/m$ is the reduced mass, and E is the total orbital energy (henceforth we use units in which $G = c = 1$). A circular orbit corresponds to $e = 0$, with $r = a =$ constant, $\Omega^2 = m/a^3$, $E/\mu = a^2\Omega^2/2 - m/a = -(m\Omega)^{2/3}/2$, and $J/\mu = \sqrt{ma} = (m/\Omega)^{1/3}$. However, if we demand only that the orbit be at apocenter, so that $\dot r = 0$ only, we have $\phi = \omega + \pi$, $r = p/(1-e)$, $\Omega^2 = (m/p^3)(1-e)^4$, and

$$\begin{aligned} E/\mu &= -\frac{1}{2}(1-e^2)\left[\frac{m\Omega_a}{(1-e)^2}\right]^{2/3}\,, \\ J/\mu m &= \left[\frac{m\Omega_a}{(1-e)^2}\right]^{-1/3}\,. \end{aligned} \tag{2}$$

where Ω_a is the angular velocity at apocenter. The energy and angular momentum expressions when the orbit is at pericenter may be obtained by letting $\Omega_a \to \Omega_p$ and $e \to -e$.

At 1PN order, the equations of motion can be solved using either the direct approach of Wagoner and Will [13], or the "perturbed osculating orbit elements approach" of Lincoln and Will [14] to yield the orbit equations

$$\begin{aligned} \tilde p/r &= 1 + \tilde e\cos(\phi-\omega) \\ &\quad + \tilde\zeta\big[-(3-\eta) + (1+9\eta/4)\tilde e^2 \\ &\quad + (7/2-\eta)\tilde e\cos(\phi-\omega) + 3\tilde e\phi\sin(\phi-\omega) \\ &\quad - (\eta/4)\tilde e^2\cos(2\phi-2\omega)\big] + O(\tilde\zeta)^2\,, \\ r^2\Omega &= (m\tilde p)^{1/2}[1 - \tilde\zeta(4-2\eta)\tilde e\cos(\phi-\omega) + O(\tilde\zeta)^2]\,, \\ E &= \mu(\dot r^2 + r^2\Omega^2)/2 - \mu m/r \\ &\quad + \frac{3}{8}\mu v^4(1-3\eta) + \frac{1}{2}\mu(m/r)^2 + \frac{1}{2}\mu\eta\dot r^2 m/r \\ &\quad + \frac{3}{2}\mu(v^2 m/r)(1+\eta/3) + O(m/r)^4\,, \end{aligned}$$

$$J = \mu|\mathbf{x}\times\mathbf{v}|\left\{1+\frac{1}{2}v^2(1-3\eta)+\frac{m}{r}(3+\eta) +O(m/r)^2\right\}, \qquad (3)$$

where $v^2 = \dot{r}^2 + r^2\Omega^2$, $\eta = \mu/m$, and $\tilde{\zeta} = m/\tilde{p}$. The limit $\tilde{e} \to 0$ corresponds to a circular PN orbit. However, at higher PN orders, neither the orbital eccentricity $\tilde{e}$ nor the semilatus rectum $\tilde{p}$ is uniquely or invariantly defined. One definition of eccentricity used by Lincoln and Will [14] was that of a Newtonian orbit momentarily tangent to the true orbit (the "osculating" eccentricity); other authors [15] define multiple "eccentricities" to encapsulate different aspects of non-circular orbits at PN order.

We define an alternative eccentricity and semilatus rectum according to:

$$e \equiv \frac{\sqrt{\Omega_p}-\sqrt{\Omega_a}}{\sqrt{\Omega_p}+\sqrt{\Omega_a}},$$
$$\zeta \equiv \frac{m}{p} \equiv \left(\frac{\sqrt{m\Omega_p}+\sqrt{m\Omega_a}}{2}\right)^{4/3}, \qquad (4)$$

where Ω_p is the value of Ω where it passes through a local maximum (pericenter), and Ω_a is the value of Ω where it passes through the *next* local minimum (apocenter). These definitions have the following virtues: (1) they reduce precisely to the normal eccentricity e and semilatus rectum p in the Newtonian limit, as can be seen from Eqs. (1); (2) they are constant in the absence of radiation reaction; (3) they are somewhat more directly connected to measurable quantities, since Ω is the angular velocity as seen from infinity (eg. as measured in the gravitational-wave signal) and one calculates only maximum and minimum values, without concern for the coordinate location in the orbit; and (4) they are straightforward to calculate in a numerical simulation of orbits without resorting to complicated definitions of "distance" between bodies. They have the defect that, when radiation reaction is included, they are not local, continuously evolving variables, but rather are some kind of orbit-averaged quantities (for this reason, they may not be as "covariant" as they seem – this issue is under investigation). Nevertheless, when an eccentric orbit decays and circularizes under radiation reaction the definition of e has the virtue that it tends naturally to zero when the orbital frequency turns from ocillatory behavior to monotonically increasing behavior (i.e. the maxima and minima merge).

By virtue of these definitions, ζ has the further property that

$$\zeta = \left(\frac{m\Omega_p}{(1+e)^2}\right)^{2/3} = \left(\frac{m\Omega_a}{(1-e)^2}\right)^{2/3}. \qquad (5)$$

It is then simple to show that the relation between e and ζ and the corresponding quantities used in the Wagoner-Will solution (3) is $e = \tilde{e}\{1+\tilde{\zeta}[9/2-\eta+(1-3\eta)\tilde{e}^2]\}$, and $\zeta = \tilde{\zeta}\{1-4\tilde{\zeta}[1-\eta/3+(1/3-\eta)\tilde{e}^2]\}$. Applying these definitions to Eqs. (3), we find the 1PN expression for $E(\Omega)$ and $J(\Omega)$ for a non-circular orbit, expressed in terms of Ω at the apocenter or pericenter:

$$\frac{E}{\mu} = -\frac{1}{2}\left(1-e^2\right)\zeta + \left[\frac{3}{8}+\frac{\eta}{24}+\left(-\frac{5}{12}+\frac{\eta}{12}\right)e^2 + \left(\frac{1}{24}-\frac{\eta}{8}\right)e^4\right]\zeta^2,$$
$$\frac{J}{\mu m} = \frac{1}{\sqrt{\zeta}}\left\{1+\left[\frac{3}{2}+\frac{1}{6}\eta-\left(\frac{1}{6}-\frac{1}{2}\eta\right)e^2\right]\zeta\right\}. \qquad (6)$$

Note that in the circular orbit limit ($e \to 0$), E and J satisfy the expected relation $dE/d\Omega = \Omega dJ/d\Omega$. We have extended this result to 3PN order using the 3PN orbit equations of Blanchet and Faye [7], using both harmonic-gauge and ADM gauge expressions, but neglecting radiation-reaction terms at 2.5PN order (details will be given in a future publication).

For those numerical simulations in which the bodies are assumed to be corotating, we must include a number of rotational effects. First is the energy of rotation of the individual corotating bodies, of order $\delta E_{\rm rot}/\mu \sim I\Omega^2/\mu \sim (m/r)(R/r)^2$, where R is the characteristic size of the body. For compact bodies, $R \sim M$, so these effects are equivalent to 2PN order and higher. For black holes, we will use the standard procedure of splitting the mass of each body into its irreducible mass and its rotational energy using the Kerr formulae $M = M_{\rm irr}/[1-4(M_{\rm irr}\Omega)^2]^{1/2}$ and $S = 4M_{\rm irr}^3\Omega/[1-4(M_{\rm irr}\Omega)^2]^{1/2}$ [11, 12]. Also, since the sequences of numerical simulations hold $m_{\rm irr} = (M_{\rm irr})_1 + (M_{\rm irr})_2$ fixed as they vary Ω, we will expand the masses that appear in the Newtonian orbital contribution to the energy and angular momentum about $m_{\rm irr}$. These contributions together yield

$$\delta E_{\rm rot}/\mu = (2/\eta)(1-3\eta)(m_{\rm irr}\Omega)^2 +(6/\eta)(1-5\eta+5\eta^2)(m_{\rm irr}\Omega)^4 -(1-e^2)(\frac{2}{3}-\eta)\zeta(m_{\rm irr}\Omega)^2,$$
$$\delta J_{\rm rot}/\mu m_{\rm irr} = (4/\eta)(1-3\eta)(m_{\rm irr}\Omega) +(8/\eta)(1-5\eta+5\eta^2)(m_{\rm irr}\Omega)^3 +\frac{1}{\sqrt{\zeta}}(m_{\rm irr}\Omega)^2(\frac{4}{3}-2\eta), \qquad (7)$$

where henceforth, $\zeta = [m_{\rm irr}\Omega_a/(1-e)^2]^{2/3}$, and μ and η are expressed in terms of irreducible masses. The first two terms in each of Eqs. (7) are the rotational terms, expanded to second order, while the third comes from expanding the Newtonian orbital term.

Similarly spin-orbit and spin-spin effects are of order $\delta E_{\rm S.O.}/\mu \sim LS/\mu r^3$ and $\delta E_{\rm S.S.}/\mu \sim S_1S_2/\mu r^3$ where L denotes the orbital angular momentum and S denotes the bodies' spin. For co-rotating bodies, these effects can be shown to be of order $\delta E_{\rm S.O.}/\mu \sim (m/r)(mR^2/r^3)$ and $\delta E_{\rm S.S.}/\mu \sim (m/r)(mR^4/r^5)$, which for compact bodies are equivalent to 3PN and 5PN order, respectively. Henceforth, we will ignore the spin-spin terms. Assuming co-rotating bodies with spins aligned with the orbital

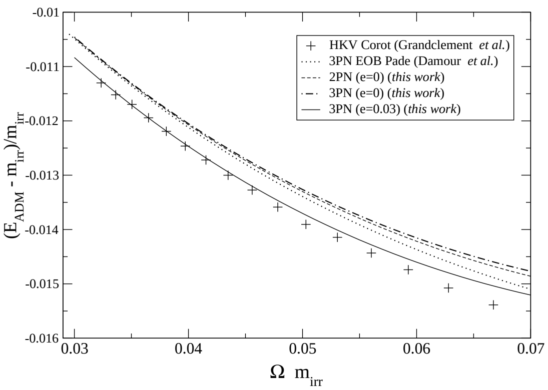

FIG. 1: Binding energy of equal-mass, co-rotating black holes vs. angular frequency.

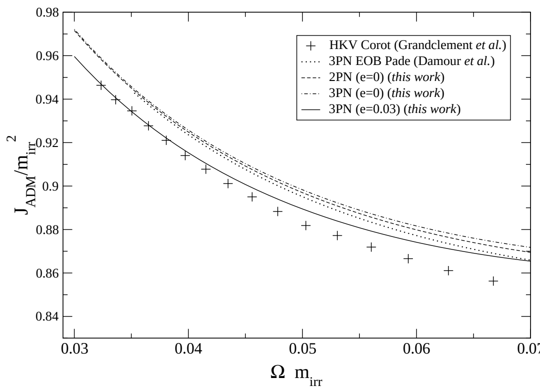

FIG. 2: Angular momentum of equal-mass, co-rotating black holes vs. angular frequency.

angular momentum, including both the direct and orbital effects of the spin-orbit terms [16, 17] as well as their effect on our definitions of e and ζ, we find

$$\begin{aligned}\frac{\delta E_{\rm S.O.}}{\mu} &= \left(-\frac{16}{3}+12\eta+\frac{4}{3}(5-11\eta)e^2-\frac{4}{3}(1-2\eta)e^4\right)\\ &\qquad\times(m_{\rm irr}\Omega)\zeta^{5/2}\,.\\ \frac{\delta J_{\rm S.O.}}{\mu m_{\rm irr}} &= \left(-\frac{40}{3}+30\eta-\frac{2}{3}(4-11\eta)e^2\right)(m_{\rm irr}\Omega)\zeta\,. \qquad (8)\end{aligned}$$

Apart from the generalization to eccentric orbits, these methods parallel exactly those of Blanchet [11].

For tidal interactions, the contribution to the orbital energy is of order $\delta E_{\rm tidal}/\mu \sim (m/r)(R/r)^5$. For black holes, tidal effects are thus equivalent to 5PN order, and will be neglected. For neutron stars, $R\sim 5M$, and tidal effects must be included, however it is sufficient to use Newtonian gravity to calculate them. These will be discussed elsewhere.

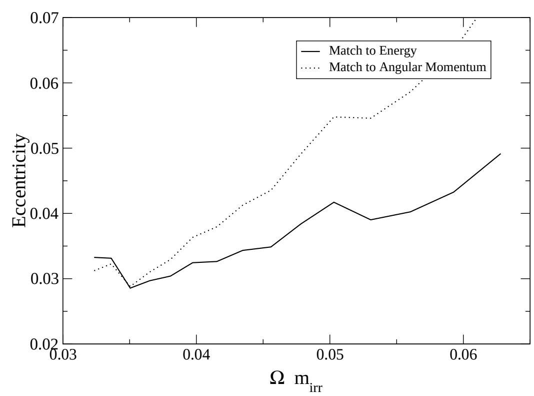

FIG. 3: Values of eccentricity giving a match to numerical HKV simulations.

III. COMPARISON WITH NUMERICAL SIMULATIONS OF COROTATING BINARY BLACK HOLES

We combine the PN, rotational and spin-orbit contributions to E and J, set $\Omega_a=\Omega$ so that the stars corotate at the orbital angular frequency at apocenter, set $\eta=1/4$ for equal-mass black holes, and plot $(E-m_{\rm irr})/m_{\rm irr}$ and $J/m_{\rm irr}^2$ versus $m_{\rm irr}\Omega$, where $m_{\rm irr}$ denotes the total irreducible mass. The results are shown in Figures 1 and 2. For $e=0$, we show results both from the 3PN orbital expressions, and from truncated 2PN orbital expressions, while for $e=0.03$ we plot the full 3PN results. The crosses denote the corresponding numerical data of Grandclément et al.[10]; the other curve is the 3PN resummation result of Damour et al.[12] for circular orbits. We note that all circular-orbit PN estimates, including conventional 2PN and 3PN approaches [11], Padé and resummation approaches [12], and our approach, agree well among themselves, but are systematically displaced from the numerical points, both for E and for J. This can also be seen in Figures 5 and 6 of [12]. By contrast, with $e\approx 0.03$, our PN estimate fits the curves perfectly for small values of $m_{\rm irr}\Omega$. This fit is merely illustrative: if the numerical simulations do have some eccentricity, there is no reason why the eccentricity should be the same for each numerical point. Figure 3 shows the values of e that best fit the numerical data over the range of Ω considered.

The conclusions are not substantially different if we use the Wagoner-Will type eccentricity $\tilde e$. Whatever definition one uses, $e=0$ gives a worse fit than a finite eccentricity.

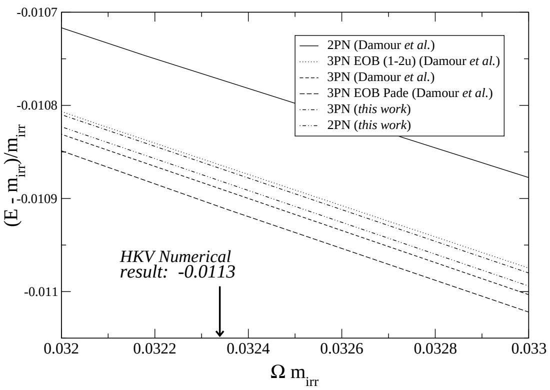

FIG. 4: Comparison of PN calculations of energy for widely separated black holes.

IV. DISCUSSION

High-order post-Newtonian approximations for circular orbits appear to give results for the energy and angular momentum of corotating binary black holes well away from the innermost orbit that are in excellent agreement – with each other. This is illustrated in Figure 4, which plots the energy from four PN results quoted by Damour et al [12], and from our 2PN and 3PN results, in the range around $\Omega m_{\rm irr} \sim 0.32$. Apart from the one 2PN result quoted by Damour et al., all are within 0.5 percent of each other. This could be viewed as an estimate of the accuracy of the PN approximation in this regime. But all results are four percent displaced from the numerical HKV data. In the absence of a quantitative estimate of the accuracy of the numerical simulations in this regime, it is difficult to decide if this difference is a signal of a physical effect, such as the small eccentricity suggested by our work.

One way to test whether the orbits represented by the HKV numerical simulations are really eccentric would be to evolve the orbits numerically for a short period of time, say 1/4 of an orbit ($\delta\phi = \pi/2$). For a nearly circular orbit of equal mass bodies, gravitational radiation reaction should cause the parameter $\zeta = m/p = (m\Omega)^{2/3}$ to evolve according to $d\zeta/d\phi = (16/5)\zeta^{7/2}$ (see, for example Eq. (3.11a) of [14]). Thus, over a quarter of an orbit, ζ should change by approximately $\delta\zeta/\zeta \approx (8\pi/5)\zeta^{5/2} \approx (8\pi/5)(m\Omega)^{5/3} \approx 0.016$, for $m\Omega \sim 0.032$. Hence the orbital separation should decrease by about 1.6 percent or the angular velocity should increase by 2.4 percent. By contrast, if the orbit has an eccentricity of 0.03 and is at apocenter, then in a quarter of an orbit, the separation should decrease by 3 percent, while the angular velocity should increase by 6 percent. Even with radiation damping, the orbit should pass through a distinct pericenter when $\delta\phi \approx \pi$, and the orbital separation should increase, while the angular velocity decreases.

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